1. (20 pts) Solve the least squares problem with $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \\ 2 & 2 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 5 \\ 6 \end{pmatrix}.$$ 

What is the minimum value of this least square problem.

2. (20 pts) Use the monomial approach to construct the lowest degree polynomial $P(x)$ that interpolates a function $f(x)$ such that

$$f(0) = 1, \quad f'(\frac{1}{2}) = 0, \quad f(-1) = 2.$$ 

Obtain an approximation for $f(\frac{1}{2})$ using this polynomial.

3. (20 pts) Construct the Lagrange interpolating polynomial for $\sqrt{x}$ at $x = 1, 2, 3$. You do not need to simplify the expression.

4. (20 pts) Suppose $P(x)$ is the lowest degree interpolating polynomial for the function $\frac{1}{x}$ at $x = 1, \frac{3}{2}, 2$. Give an upper bound for the magnitude of the error $|P(x) - \frac{1}{x}|$ at $x = \frac{5}{4}$ and $x = \frac{5}{2}$.

5. (20 pts) The Lagrange form of the interpolating polynomial is

$$P_n(x) = \sum_{j=0}^{n} f(x_j)L_j(x),$$

where $L_j(x)$ are the fundamental polynomials. Show that

$$\sum_{j=0}^{n} L_j(x) = 1.$$ 

6. (20 pts) Form the Newton polynomial of degree 2 which interpolates the following data: $(x_1, f(x_1)) = (0, 1)$, $(x_2, f(x_2)) = (1, 4)$, and $(x_3, f(x_3)) = (2, 15)$. Give a bound on the interpolation error at the point $x = \frac{1}{2}$ assuming that the third derivative of $f$ is less than or equal to 6.
7. (20 pts) (Exercise 5 on page 99 in Textbook)
Obtain the approximation of function $f$ using natural cubic splines for $f(x)$ whose values are given at three points:

$$f(0) = 1, \quad f(1) = 6, \quad f(2) = 5, \quad f(3) = 4.$$ 

8. (20 pts) Matlab.
Given a linear system, $Ax = b$, where

$$A = \begin{pmatrix}
    10 & -1 & 2 & 0 \\
    -1 & 11 & -1 & 3 \\
    2 & -1 & 10 & 0 \\
    0 & 3 & -1 & 8
\end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix}
    6 \\
    25 \\
    -11 \\
    -11
\end{pmatrix}$$

Consider the Jacobi and Gauss-Seidel iterative methods for this linear system with Euclidean norm. Suppose $x^{(0)} = 0$, the tolerance $TOL = 10^{-2}$, and the maximum number of iterations, $MAX = 25$.

(a) For the Jacobi method, write a function, $\text{function } X = \text{jacobi}(A,b,x0,TOL,MAX)$, which outputs an approximation vector $X$ to $x$ and prints the error and relative error in each iteration in a table as

<table>
<thead>
<tr>
<th>iteration</th>
<th>error</th>
<th>rel. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Use the function, jacobi(), to solve the linear system.

(b) Similarly, write a function, $\text{function } X = \text{gseid}(A,b,x0,TOL,MAX)$, for the Gauss-Seidel method. Solve the linear system by the function, gseid().

(c) Solve problem 2.1 on p.115 in the textbook with your functions and compare your results with the solutions in the textbook.

9. (20 pts) Matlab.
Fit a cubic spline and a fifth-degree polynomial to the following data:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-30</td>
</tr>
<tr>
<td>4</td>
<td>-40</td>
</tr>
<tr>
<td>5</td>
<td>-50</td>
</tr>
</tbody>
</table>

Plot the data point, spline and the polynomial on the same graph. Which curve appears to give the more realistic representation of any underlying function from which the data might have been taken?

You should write your own functions for cubic spline and polynomial interpolations. For the fifth-degree polynomial, please use Lagrange Interpolation.