1. Find an dfa that accepts the following language

\[ L(ab^*a^*) \cap L(a^*b^*a) . \]

Please also write the corresponding regular expression.

**Answer:**
The dfa’s \( M_1 = (Q, \Sigma, \delta_1, q_0, F_1) \) and \( M_2 = (P, \Sigma, \delta_2, p_0, F_2) \) for \( L(ab^*a^*) \) and \( L(a^*b^*a) \) are as below.

![DFAs](image)

Then, we can use the constructive proof used in Theorem 4.1 to derive a dfa \( M' \) for

\[ L(ab^*a^*) \cap L(a^*b^*a) . \]

Let \( M' = M_1 \cap M_2 = (Q \times P, \Sigma, \delta', (q_0, p_0), F') \) where the transition function \( \delta' \) is defined as follows.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( (q_0, p_0) )</th>
<th>( (q_1, p_1) )</th>
<th>( (q_2, p_1) )</th>
<th>( (q_1, p_2) )</th>
<th>( (q_2, p_2) )</th>
<th>( (q_2, p_3) )</th>
<th>( (q_2, p_4) )</th>
<th>( (q_3, p_2) )</th>
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<tr>
<td>( b )</td>
<td>( (q_1, p_1) )</td>
<td>( (q_2, p_1) )</td>
<td>( (q_2, p_2) )</td>
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Note that, the final states in \( M' \) are \( (q_2, p_1) \) and \( (q_2, p_3) \). The transition graph of \( M' \) is given below. The corresponding regular expression of \( L(M') \) is \( a(b^*|a^*)a. \)
2. (10 pts) Let $L_1 = L(a^*baa^*)$ and $L_2 = L(aba^*)$. Find $L_1 / L_2$

**Answer:** 
$L_1 / L_2 = L(a^*)$.

3. (20 pts) The tail of a language $L$ is defined as the set of all suffixes of its strings, that is 

$$tail(L) = \{ y : xy \in L \text{ for some } x \in \Sigma^* \}.$$

Show that if $L$ is regular, so is $tail(L)$.

**Answer:**
This problem is straightforward. You need to create a finite automaton $M'$ for the tail of the given language $L$. Suppose there is an nfa $M = (Q, \Sigma, \delta, q_0, F)$ accepting $L$. To construct $M'$, we consider the set of all states with $\delta(q,w) \in F$ for some $w$. Add a new initial state and $\lambda$-transition function from it to all the above states to $M$ to derive $M'$. Then, you need to show that $w \in tail(L)$ if and only if $M'$ accepts $w$.

4. (20 pts) Show that there exists an algorithm that can determine for every regular language $L$, whether or not $|L| \geq 5$.

**Answer:** Look at the dfa of $L$. If there is a cycle, then $|L| \geq 5$. Otherwise, check the number of all possible paths ending with a final states.

5. (20 pts) Let $L = \{ w : n_a(w) = n_b(w) \}$. Please determine that $L^*$ is regular or not? Please provide your arguments.

**Answer:**
Pick $\{a^m b^m \}$ as starting string. We can see $L$ is not regular. Also, here $L = L^*$. Hence, $L^*$ is not regular.

6. (20 pts) Prove that the following languages are not regular.

   (a) $L = \{a^n b^l a^k : k \geq n + l \}$.
   (b) $L = \{a^n b^l a^k : k \neq n + l \}$.
   (c) $L = \{a^n b^l a^k : n = l \text{ or } l \neq k \}$.
   (d) $L = \{ w : n_a(w) \neq n_b(w) \}$.
   (e) $L = \{ w w : w \in \{ a, b \}^* \}$.
   (f) $L = \{ w w w w^R : w \in \{ a, b \}^* \}$.

**Answer:**

   (a) $L = \{a^n b^l a^k : k \geq n + l \}$. Given $m$, we pick $w = a^m b^m a^{2m}$ with $|w| \geq m$. Let $w = xyz$ where $|xy| \leq m$ and $|y| \geq 1$. Then, $y = a^k$, where $1 \leq k \leq m$. Now, we consider the pumping strings 

$$w_i = a^{m+(i-1)k} b^m a^{2m}.$$

If $i \geq 2$, then $m + (i - 1)k > m$. So, $w_i \notin L$. According to the Pumping Lemma, $L$ is not regular.
(b) This can be done by first solving that \( \mathcal{L} = \{ a^n b^k a^k : k = n + l \} \) is not regular as in the previous problem. Then \( L \) is not regular; otherwise \( \mathcal{L} \) is regular. The other way to show this is to apply the pumping lemma directly.

Given \( m \). We pick \( w = a^m b^m a^{2m+1} \) with \( |w| \geq m \). Consider \( w = xyz \) where \( |xy| \leq m \) and \( |y| \geq 1 \). Then, \( y = a^k \), where \( 1 \leq k \leq m \). Now, we choose \( y = a \). consider the pumping strings

\[ w_i = a^{m+(i-1)b^m} a^{2m+1}. \]

When \( i = 2 \), \( w_2 = a^m b^m a^{2m+1} \notin L \). According to the Pumping Lemma, \( L \) is not regular.

(c) \( L = \{ a^n b^k a^k : n = l \ or \ l \neq k \} \). Given \( m \), we pick \( w = a^m b^m a^m \) with \( |w| \geq m \). Let \( w = xyz \) where \( |xy| \leq m \) and \( |y| \geq 1 \). Then, \( y = a^k \), where \( 1 \leq k \leq m \). Now, we consider the pumping strings

\[ w_i = a^{m+(i-1)k} b^m a^m. \]

When \( i = 0 \), \( w_0 = a^{m-k} b^m a^m \notin L \). According to the Pumping Lemma, \( L \) is not regular.

(d) If \( L = \{ w : n_a(w) \neq n_b(w) \} \) is regular, then \( \mathcal{L} = \{ w : n_a(w) = n_b(w) \} \) is regular. Note that \( L(a^* b^*) \cap \mathcal{L} = L(a^* b^n) \). If we want to prove \( L = \{ w : n_a(w) \neq n_b(w) \} \) is not regular, we only need to prove \( L(a^n b^n) \) is not regular. This will imply that \( L = \{ w : n_a(w) \neq n_b(w) \} \) is not regular.

Given \( m \), we pick \( w = a^m b^m \) with \( |w| \geq m \). Let \( w = xyz \) where \( |xy| \leq m \) and \( |y| \geq 1 \). Then, \( y = a^k \), where \( 1 \leq k \leq m \). Now, we consider the pumping strings

\[ w_i = a^{m+(i-1)k} b^m. \]

When \( i = 0 \), \( w_0 = a^{m-k} b^m \notin L(a^n b^n) \). According to the Pumping Lemma, \( L(a^n b^n) \) is not regular. Hence, \( L = \{ w : n_a(w) \neq n_b(w) \} \) is not regular.

(e) Given \( m \), we pick \( w = a^m b^m a^m b^m \) with \( |w| \geq m \). Let \( w = xyz \) where \( |xy| \leq m \) and \( |y| \geq 1 \). Then, \( y = a^k \), where \( 1 \leq k \leq m \). Now, we consider the pumping strings

\[ w_i = a^{m+(i-1)k} b^m a^m b^m. \]

When \( i = 0 \), \( w_0 = a^{m-k} b^m a^m b^m \notin L \). According to the Pumping Lemma, \( L \) is not regular.

(f) Given \( m \), we pick \( w = a^m b^m a^m b^m a^m b^m a^m \) with \( |w| \geq m \). Let \( w = xyz \) where \( |xy| \leq m \) and \( |y| \geq 1 \). Then, \( y = a^k \), where \( 1 \leq k \leq m \). Now, we consider the pumping strings

\[ w_i = a^{m+(i-1)k} b^m a^m b^m a^m b^m a^m. \]

When \( i = 0 \), \( w_0 = a^{m-k} b^m a^m b^m a^m b^m a^m \notin L \). According to the Pumping Lemma, \( L \) is not regular.

7. (20 pts) Consider the language

\[ L = \{ a^n : n \text{ is not a perfect square} \} \]

(a) Show that this language is not regular by applying the pumping lemma directly.
(b) Then show the same thing by using the closure properties of regular languages.

**Answer:**

(a) We have to pick a very special string $a^{M^2+1}$ to start with. Suppose the middle string has length $k$; then the pumped string is $a^{M^2+1+(i-1)k}$. If we now pick $M = m!$, then $i$ can be chosen so that $M^2 + 1 + (i - 1)k = (M + 1)^2$.

(b) $\mathcal{L}$ is not regular, so does $L$.

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