Three springs and a mass are attached to a rigid, weightless bar $PQ$ as shown in Fig. 2.40. Find the natural frequency of vibration of the system.

![Diagram of a system with three springs and a mass attached to a rigid, weightless bar $PQ$.]

**FIGURE 2.40**

For small angular rotation of bar $PQ$ about $P$,

\[
\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2
\]

i.e.,

\[
(k_{12})_{eq} = \frac{(k_1 l_1^2 + k_2 l_2^2)}{l_3^2}
\]

Let $k_{eq}$ = overall spring constant at $Q$.

\[
\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}
\]

\[
k_{eq} = \frac{(k_{12})_{eq}}{(k_{12})_{eq} + k_3} = \frac{k_1 \left(\frac{l_1}{l_3}\right)^2 + k_2 \left(\frac{l_2}{l_3}\right)^2}{k_1 \left(\frac{l_1}{l_3}\right)^2 + k_2 \left(\frac{l_2}{l_3}\right)^2 + k_3}
\]

\[
\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m \left( k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2 \right)}}
\]
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\[ x_1 = \frac{2W}{2k} \]

\[ x_2 = \frac{2W}{2k} \]

\[ x = 2x_1 + 2x_2 = 2\frac{W}{k} + 2\frac{W}{k} = 4\frac{W}{k} \]

\[ k_{eq} = \frac{W}{x} = \frac{W}{4W/k} = \frac{k}{4} \]

\[ \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}} \]
\[ x_1 = \frac{\delta}{2} \]
\[ x_2 = \frac{x_1}{2} \]
\[ x = \frac{x_2}{2} \]

\[ \delta = \frac{W}{8k} \quad \text{spring deformation} \]

\[ x = \frac{\delta}{8} = \frac{W}{64k} \]

\[ k_{eq} = \frac{W}{x} = \frac{W}{W/64k} = 64k \]

\[ \omega_n = \sqrt{\frac{k_{eq}}{m}} = 8\sqrt{\frac{k}{m}} \]
2.40 A helical spring of stiffness $k$ is cut into two halves and a mass $m$ is connected to the two halves as shown in Fig. 2.70(a). The natural time period of this system is found to be 0.5 s. If an identical spring is cut so that one part is one-fourth and the other part three-fourths of the original length, and the mass $m$ is connected to the two parts as shown in Fig. 2.70(b), what would be the natural period of the system?

\[
m \ddot{x} + (k_1 x + k_2 x) = 0
\]

\[
k = \frac{d^4 G}{8 D^3 n} \propto \frac{1}{n} \propto \frac{1}{\ell}
\]

\[
\frac{k_{\ell/2}}{k} = \frac{\ell}{\ell/2} = 2, \quad \therefore k_{\ell/2} = 2k, \quad k_{\ell/4} = 4k, \quad k_{3\ell/4} = \frac{4}{3}k
\]
\[ \tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \]

\[ 0.5 = 2\pi \sqrt{\frac{m}{4k}} \]

\[ \sqrt{\frac{m}{k}} = \frac{1}{2\pi} \]

\[ k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k \]

\[ \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{16k}{3m}} \]

\[ \tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \quad \text{where} \quad k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k \]

\[ \therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi}{4} \sqrt{\frac{\sqrt{m}}{k}} = \frac{2\pi}{4} \left( \frac{1}{2\pi} \right) = 0.4330 \text{ sec} \]
The free vibration responses of an electric motor of weight 500 N mounted on different types of foundations are shown in Figs. 2.91 (a) and (b). Identify the following in each case: (i) the nature of damping provided by the foundation, (ii) the spring constant and damping coefficient of the foundation, and (iii) the undamped and damped natural frequencies of the electric motor.

\[ \tau_d = 0.2 \text{ sec}, \ f_d = 5 \text{ Hz}, \ \omega_d = 31.416 \text{ rad/sec}. \]

\[
\ln \left( \frac{x_i}{x_{i+1}} \right) = \ln 2 = 0.6931 = \frac{2 \pi \zeta}{\sqrt{1 - \zeta^2}}
\]
or
\[ 39.9590 \zeta^2 = 0.4804 \quad \text{or} \quad \zeta = 0.1096 \]

Since \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \), we find

\[ \omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec} \]

\[ k = m \omega_n^2 = \left( \frac{500}{9.81} \right) (31.6065)^2 = 5.0916 \times 10^4 \text{ N/m} \]

\[ \zeta = \frac{c}{c_c} = \frac{c}{2 \ m \ \omega_n} \]

Hence \( c = 2 \ m \ \omega_n \zeta = 2 \left( \frac{500}{9.81} \right) (31.6065) (0.1096) = 353.1164 \text{ N\text{-}s/m} \)