Phase Plane Analysis

Phase plane analysis is one of the most important techniques for studying the behavior of nonlinear systems, since there is usually no analytical solution for a nonlinear system.
Background

- The response characteristics (relative speed of response) for unforced systems were dependent on the initial conditions.

- Eigenvalue/eigenvector analysis allowed us to predict the fast and slow (or stable and unstable) initial conditions.

- Another way of obtaining a feel for the effect of initial conditions is to use a phase-plane plot.

- A phase-plane plot for a two-state variable system consists of curves of one state variable versus the other state variable ($x_1(t)$ vs. $x_2(t)$), where each curve is based on a different initial condition.
Example 1: A Stable Equilibrium Point (Node Sink)

- Consider the system

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= -4x_2
\end{align*}
\]

\[\Rightarrow \quad x_1(t) = x_1(0) e^{-t}, \quad x_2(t) = x_2(0) e^{-4t}\]

- Plot \(x_1\) and \(x_2\) as a function of time for a large number of initial conditions

The solutions converge to \((0,0)\) for all initial conditions.

The point \((0,0)\) is a stable equilibrium point for the system.

\(\Rightarrow\) **stable node**
**Phase-Plane Behavior of Linear Systems**

- **Example 2: An Unstable Equilibrium Point (Saddle)**
  - Consider the system
    \[
    \dot{x}_1 = -x_1 \quad \Rightarrow \quad x_1(t) = x_1(0)e^{-t}
    \]
    \[
    \dot{x}_2 = 4x_2 \quad \Rightarrow \quad x_2(t) = x_2(0)e^{4t}
    \]
  - Plot \(x_1\) and \(x_2\) as a function of time for a large number of initial conditions

If the initial condition for \(x_2\) was 0, then the trajectory reached the origin. Otherwise, the solution will always leave the origin.

The point (0,0) is an unstable equilibrium point for the system

⇒ saddle point

The \(x_1\) axis represents a stable subspace and the \(x_2\) axis represents an unstable subspace.
**Phase-Plane Behavior of Linear Systems**

- **Example 3: Another Saddle Point Problem**
  - Consider the system
    \[
    \begin{align*}
    \dot{x}_1 &= 2x_1 + x_2 \\
    \dot{x}_2 &= 2x_1 - x_2
    \end{align*}
    \]
    \[\Rightarrow \quad \dot{x} = Ax \quad A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\]
  - Eigenvalues
    \[\lambda_1 = -1.5616 \quad \lambda_2 = 2.5616\]
  - Eigenvectors
    \[\xi_1 = \begin{bmatrix} 0.2703 \\ -0.9628 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0.8719 \\ 0.4896 \end{bmatrix} \quad \text{(unstable subspace)}\]
    \[\text{(stable subspace)}\]
Phase-Plane Behavior of Linear Systems

- **Example 4: Unstable Focus (Spiral Source)**
  - Consider the system
    \[
    \begin{align*}
    \dot{x}_1 &= x_1 + 2x_2 \\
    \dot{x}_2 &= -2x_1 + x_2
    \end{align*}
    \Rightarrow \quad \dot{x} = Ax
    \]
    \[
    A = \begin{bmatrix}
    1 & 2 \\
    -2 & 1
    \end{bmatrix}
    \]
  - Eigenvalues: \(1 \pm 2i\) (unstable system because the real part is positive)
Phase-Plane Behavior of Linear Systems

• Example 5: **Center**
  
  – Consider the system
    \[
    \begin{align*}
    \dot{x}_1 &= -x_1 - x_2 \\
    \dot{x}_2 &= 4x_1 + x_2
    \end{align*}
    \Rightarrow \quad \dot{x} = Ax
    \]
    
  – Eigenvalues \( = \pm 1.7321i \)

  Since the real part of the eigenvalues is zero, there is a periodic solution, resulting a phase-plane plot where the equilibrium point is a **center**.
Generalization of Phase-Plane Behavior

• Consider second-order systems

\[ \dot{x} = Ax \]

Jacobian matrix \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \)

• Eigenvalues of \( A \) \( \Rightarrow \) \( \det (\lambda I - A) = 0 \)

\[
\det (\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0
\]

\[
\lambda = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}(A))^2 - 4\det(A)}}{2}
\]

• Phase-plane behavior resulting from \( \lambda_1 \) and \( \lambda_2 \)
  
  – Sinks (stable nodes) : \( \Re(\lambda_1) < 0 \) and \( \Re(\lambda_2) < 0 \)
  
  – Saddles (unstable) : \( \Re(\lambda_1) < 0 \) and \( \Re(\lambda_2) > 0 \)
  
  – Sources (unstable nodes) : \( \Re(\lambda_1) > 0 \) and \( \Re(\lambda_2) > 0 \)
  
  – Spirals : \( \lambda_1 \) and \( \lambda_2 \) are complex conjugate. If \( \Re(\lambda_1) < 0 \) then stable, if \( \Re(\lambda_1) > 0 \) then unstable
• Dynamic behavior diagram for second-order linear systems

The eigenvalue will be complex if

\[ 4 \det(A) > (\text{tr}(A))^2 \]
• Phase-plane behavior as a function of eigenvalue location
Nonlinear Systems

• Nonlinear systems will often have the same general phase-plane behavior as the model linearized about the equilibrium (steady-state) point, when the system is close to that particular equilibrium point.

• Nonlinear systems often have multiple steady-state solutions. Phase-plane analysis of nonlinear systems provides an understanding of which steady-state solution that a particular set of initial conditions will converge to.

• The local behavior (close to one of the steady-state solutions) can be understood from a linear phase-plane analysis of the particular steady-state solution (equilibrium point).
Nonlinear System Example

• Consider the system

\[
\frac{dz_1}{dt} = z_2 (z_1 + 1) \\
\frac{dz_2}{dt} = z_1 (z_2 + 3)
\]

– Two steady-state (equilibrium) solutions

• Equilibrium 1: trivial \( z_{1s} = 0 \quad z_{2s} = 0 \)

• Equilibrium 2: nontrivial \( z_{1s} = -1 \quad z_{2s} = -3 \)

– Linearizing the system, we have

\[
A = \begin{bmatrix}
    z_{2s} & z_{1s} + 1 \\
    z_{2s} + 3 & z_{1s}
\end{bmatrix}
\]
• Equilibrium 1 (Trivial)  (0,0)

\[
A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \quad \lambda_1 = -\sqrt{3} \quad \lambda_2 = \sqrt{3}
\]

- This equilibrium point is a **saddle point**

- Stable eigenvector \( \xi_1 = \begin{bmatrix} -0.5 \\ 0.866 \end{bmatrix} \)  
  Unstable eigenvector \( \xi_2 = \begin{bmatrix} 0.5 \\ 0.866 \end{bmatrix} \)

- The phase-plane of the linearized model around equilibrium point 1

Linearized model:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} x
\]

where \( x = z - z_s \)
• **Equilibrium 2 (Nontrivial) (-1,-3)**

\[
A = \begin{bmatrix}
-3 & 0 \\
0 & -1
\end{bmatrix} \quad \lambda_1 = -3 \quad \lambda_2 = -1
\]

- This equilibrium point is a **stable node**

- “Fast” stable eigenvector \( \xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) “Slow” stable eigenvector \( \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

- The phase-plane of the linearized model around equilibrium point 2
• The phase-plane diagram of the nonlinear model

The linearized models capture the behavior of the nonlinear model when close to one of the equilibrium points
Exercise: Interacting tanks

- Two interacting tank in series with outlet flowrate being function of the square root of tank height
  - Parameter values
    \[ R_1 = 2.5 \frac{ft^{2.5}}{min} \quad R_2 = \frac{5}{\sqrt{6}} \frac{ft^{2.5}}{min} \quad A_1 = 5 ft^2 \quad A_2 = 10 ft^2 \]
  - Input variable \( F = 5 ft^3/min \)
  - Steady-state height values: \( h_{1s} = 10, h_{2s} = 6 \)

Perform a phase-plane analysis and discuss your results

Linearized model:

\[
A = \begin{bmatrix}
-0.125 & 0.125 \\
0.0625 & -0.1042
\end{bmatrix}
\]

\[
V = \begin{bmatrix}
-0.8466 & -0.7827 \\
0.5323 & -0.6224
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
-0.2036 & 0 \\
0 & -0.0256
\end{bmatrix}
\]
% Nonlinear Model for h10 = 8 : 12
  for h20 = 4 : 8
    y0=[h10 h20];
    [t,y]=ode45('phaseplane', [0 200], y0);
    plot(y(:,1),y(:,2), y(1,1),y(1,2),'o'); hold on;
    xlabel('h_1'); ylabel('h_2');
    title('Phase-Plane Plot');
  end
end % Linearized Model for h10 = -2 : 2
  for h20 = -2:2
    y0=[h10 h20];
    [t,y]=ode45('phaseplane_L', [0 200], y0);
    y(:,1) = y(:,1)+10; y(:,2) = y(:,2)+6;
    plot(y(:,1),y(:,2),'r', y(1,1),y(1,2),'o'); hold on;
  end
end

function dy = phaseplane(t,y)
A1=5; A2=10; R1=2.5; R2=5/sqrt(6); F=5;
dy(1) = F/A1-R1/A1*sqrt(y(1)-y(2));
dy(2) = R1/A2*sqrt(y(1)-y(2))-R2/A2*sqrt(y(2));
dy = dy';
end

function dy = phaseplane_L(t,y)
A = [-0.125 0.125; 0.0625 -0.1042];
dy = A*y;
end
Exercise: Bioreactor

- A model for a bioreactor

\[
\frac{dx_1}{dt} = (\mu - 0.4) x_1 \\
\frac{dx_2}{dt} = (4 - x_2) 0.4 - \frac{\mu x_1}{0.4} \\
\mu = \frac{0.53 x_2}{0.12 + x_2} \quad \text{(growth rate)}
\]

\(x_1: \text{biomass concentration}\)
\(x_2: \text{substrate concentration}\)

Perform a phase-plane analysis.