Linearization of Nonlinear Models

• Most chemical process models are nonlinear, but they are often linearized to perform a simulation and stability analysis.

• Linear models are easier to understand (than nonlinear models) and are necessary for most control system design methods.
Single Variable Example

• A general single variable nonlinear model

\[ \frac{dx}{dt} = f(x) \]

• The function \( f(x) \) can be approximated by a Taylor series approximation around the steady-state operating point \( (x_s) \)

\[ f(x) = f(x_s) + \frac{\partial f}{\partial x}(x - x_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x - x_s)^2 + \text{high order terms} \]

• Neglect the quadratic and higher order terms

\[ f(x) \approx f(x_s) + \frac{\partial f}{\partial x}(x - x_s) \]

At steady-state
\[ \frac{dx_s}{dt} = f(x_s) = 0 \]

The partial derivative of \( f(x) \) with respect to \( x \), evaluated at the steady-state

\[ \frac{dx}{dt} = f(x) \approx \frac{\partial f}{\partial x}(x - x_s) \]
• Since the derivative of a constant \( (x_s) \) is zero

\[
\frac{dx}{dt} = \frac{d(x - x_s)}{dt}
\]

\[
\frac{d(x - x_s)}{dt} \approx \frac{\partial f}{\partial x}igg|_{x_s} (x - x_s)
\]

• We are often interested in deviations in a state from a steady-state operating point (deviation variable)

\[
\frac{d\bar{x}}{dt} \approx \frac{\partial f}{\partial x}igg|_{x_s} \bar{x}
\]

\[
\bar{x} = x - x_s \quad \text{: the change or perturbation from a steady-state value}
\]

• Write in state-space form

\[
\frac{d\bar{x}}{dt} \approx a \bar{x} \quad \text{where} \quad a = \frac{\partial f}{\partial x}igg|_{x_s}
\]
One State Variable and One Input Variable

- Consider a function with one state variable and one input variable

\[ \dot{x} = \frac{dx}{dt} = f(x,u) \]

- Using a Taylor Series Expansion for \( f(x,u) \)

\[
\dot{x} = f(x_s, u_s) + \left. \frac{\partial f}{\partial x} \right|_{x_s,u_s} (x - x_s) + \left. \frac{\partial f}{\partial u} \right|_{x_s,u_s} (u - u_s) + \frac{1}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_s,u_s} (x - x_s)^2
\]

\[
+ \left. \frac{\partial^2 f}{\partial x \partial u} \right|_{x_s,u_s} (x - x_s)(u - u_s) + \frac{1}{2} \left. \frac{\partial^2 f}{\partial u^2} \right|_{x_s,u_s} (u - u_s)^2 + \text{high order terms}
\]

- Truncating after the linear terms

\[
\dot{x} \approx f(x_s, u_s) + \left. \frac{\partial f}{\partial x} \right|_{x_s,u_s} (x - x_s) + \left. \frac{\partial f}{\partial u} \right|_{x_s,u_s} (u - u_s)
\]

\[
\frac{d(x - x_s)}{dt} \approx \left. \frac{\partial f}{\partial x} \right|_{x_s,u_s} (x - x_s) + \left. \frac{\partial f}{\partial u} \right|_{x_s,u_s} (u - u_s)
\]
• Using deviation variables, \( \bar{x} = x - x_s \) and \( \bar{u} = u - u_s \)

\[
\frac{d\bar{x}}{dt} \approx \left. \frac{\partial f}{\partial x} \right|_{x_s, u_s} \bar{x} + \left. \frac{\partial f}{\partial u} \right|_{x_s, u_s} \bar{u}
\]

• Write in state-space form

\[
\frac{d\bar{x}}{dt} \approx a \bar{x} + b \bar{u} \quad \text{where} \quad a = \left. \frac{\partial f}{\partial x} \right|_{x_s, u_s} \quad b = \left. \frac{\partial f}{\partial u} \right|_{x_s, u_s}
\]

• If there is a single output that is a function of the state and input

\[
y = g(x, u)
\]

• Perform a Taylor series expansion and truncate high order terms

\[
g(x, u) \approx g(x_s, u_s) + \left. \frac{\partial g}{\partial x} \right|_{x_s, u_s} (x - x_s) + \left. \frac{\partial g}{\partial u} \right|_{x_s, u_s} (u - u_s)
\]

\[
y - y_s = \left. \frac{\partial g}{\partial x} \right|_{x_s, u_s} (x - x_s) + \left. \frac{\partial g}{\partial u} \right|_{x_s, u_s} (u - u_s)
\]

\[
\bar{y} = c \bar{x} + d \bar{u} \quad \text{where} \quad c = \left. \frac{\partial g}{\partial x} \right|_{x_s, u_s} \quad d = \left. \frac{\partial g}{\partial u} \right|_{x_s, u_s}
\]
Linearization of Multistate Models

- Two-state system
  \[
  \dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, x_2, u) \\
  y = g(x_1, x_2, u) \\
  \dot{x}_2 = \frac{dx_2}{dt} = f_2(x_1, x_2, u)
  \]

- Perform Taylor series expansion of the nonlinear functions and neglect high-order terms

\[
\begin{align*}
  f_1(x_1, x_2, u) &= f_1(x_{1s}, x_{2s}, u_s) + \left. \frac{\partial f_1}{\partial x_1} \right|_{x_{1s}, x_{2s}, u_s} (x_1 - x_{1s}) + \left. \frac{\partial f_1}{\partial x_2} \right|_{x_{1s}, x_{2s}, u_s} (x_2 - x_{2s}) + \left. \frac{\partial f_1}{\partial u} \right|_{x_{1s}, x_{2s}, u_s} (u - u_s) \\
  f_2(x_1, x_2, u) &= f_2(x_{1s}, x_{2s}, u_s) + \left. \frac{\partial f_2}{\partial x_1} \right|_{x_{1s}, x_{2s}, u_s} (x_1 - x_{1s}) + \left. \frac{\partial f_2}{\partial x_2} \right|_{x_{1s}, x_{2s}, u_s} (x_2 - x_{2s}) + \left. \frac{\partial f_2}{\partial u} \right|_{x_{1s}, x_{2s}, u_s} (u - u_s) \\
  g(x_1, x_2, u) &= g(x_{1s}, x_{2s}, u_s) + \left. \frac{\partial g}{\partial x_1} \right|_{x_{1s}, x_{2s}, u_s} (x_1 - x_{1s}) + \left. \frac{\partial g}{\partial x_2} \right|_{x_{1s}, x_{2s}, u_s} (x_2 - x_{2s}) + \left. \frac{\partial g}{\partial u} \right|_{x_{1s}, x_{2s}, u_s} (u - u_s)
\end{align*}
\]
• For the linearization about the steady-state

\[ f_1(x_{1s}, x_{2s}, u_s) = f_2(x_{1s}, x_{2s}, u_s) = 0 \quad g(x_{1s}, x_{2s}, u_s) = y_s \]

\[ \frac{dx_1}{dt} = \frac{d(x_1 - x_{1s})}{dt} \quad \frac{dx_2}{dt} = \frac{d(x_2 - x_{2s})}{dt} \]

• We can write the state-space model

\[
\begin{bmatrix}
\frac{d(x_1 - x_{1s})}{dt} \\
\frac{d(x_2 - x_{2s})}{dt}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}|_{x_{1s}, x_{2s}, u_s} & \frac{\partial f_1}{\partial x_2}|_{x_{1s}, x_{2s}, u_s} \\
\frac{\partial f_2}{\partial x_1}|_{x_{1s}, x_{2s}, u_s} & \frac{\partial f_2}{\partial x_2}|_{x_{1s}, x_{2s}, u_s}
\end{bmatrix} \begin{bmatrix}
x_1 - x_{1s} \\
x_2 - x_{2s}
\end{bmatrix} + \begin{bmatrix}
\frac{\partial f_1}{\partial u}|_{x_{1s}, x_{2s}, u_s} \\
\frac{\partial f_2}{\partial u}|_{x_{1s}, x_{2s}, u_s}
\end{bmatrix} [u - u_s]
\]

\[ y - y_s = \begin{bmatrix}
\frac{\partial g}{\partial x_1}|_{x_{1s}, x_{2s}, u_s} & \frac{\partial g}{\partial x_2}|_{x_{1s}, x_{2s}, u_s}
\end{bmatrix} \begin{bmatrix}
x_1 - x_{1s} \\
x_2 - x_{2s}
\end{bmatrix} + \begin{bmatrix}
\frac{\partial g}{\partial u}|_{x_{1s}, x_{2s}, u_s}
\end{bmatrix} [u - u_s]
\]

\[ \dot{x} = A \bar{x} + B \bar{u} \]
\[ \bar{y} = C \bar{x} + D \bar{u} \]
Generalization

- Consider a general nonlinear model with $n$ state variables, $m$ input variables, and $r$ output variables
  \[ \begin{align*}
  \dot{x}_1 &= f_1(x_1, \ldots, x_n, u_1, \ldots, u_m) \\
  &\vdots \\
  \dot{x}_n &= f_n(x_1, \ldots, x_n, u_1, \ldots, u_m) \\
  y_1 &= g_1(x_1, \ldots, x_n, u_1, \ldots, u_m) \\
  &\vdots \\
  y_r &= g_r(x_1, \ldots, x_n, u_1, \ldots, u_m)
  \end{align*} \]

- Vector notation:
  \[ \begin{align*}
  \dot{x} &= f(x, u) \\
  y &= g(x, u)
  \end{align*} \]

- Elements of the linearization matrices

\[
\begin{align*}
A_{ij} &= \left. \frac{\partial f_i}{\partial x_j} \right|_{x, u} \\
B_{ij} &= \left. \frac{\partial f_i}{\partial u_j} \right|_{x, u} \\
C_{ij} &= \left. \frac{\partial g_i}{\partial x_j} \right|_{x, u} \\
D_{ij} &= \left. \frac{\partial g_i}{\partial u_j} \right|_{x, u}
\end{align*} \]

- State-space form:
  \[ \begin{align*}
  \dot{x} &= A \bar{x} + B \bar{u} \\
  \bar{y} &= C \bar{x} + D \bar{u}
  \end{align*} \]

  or
  \[ \begin{align*}
  \dot{x} &= A x + B u \\
  y &= C x + D u
  \end{align*} \]

(The “overbar” is usually dropped)
Example: Interacting Tanks

- Two interacting tank in series with outlet flowrate being function of the square root of tank height

\[
F_1 = R_1 \sqrt{h_1 - h_2}
\]

\[
F_2 = R_2 \sqrt{h_2}
\]

- Modeling equations

\[
\frac{dh_1}{dt} = \frac{F - R_1 \sqrt{h_1 - h_2}}{A_1} = f_1(h_1, h_2, F)
\]

\[
\frac{dh_2}{dt} = \frac{R_1 \sqrt{h_1 - h_2} - R_2 \sqrt{h_2}}{A_2} = f_2(h_1, h_2, F)
\]
• Assume only the second tank height is measured. The output, in deviation variable form is\[ y = h_2 - h_{2s} \]

• There are two state variables, one input variable, one output variable

\[ h_s = \begin{bmatrix} h_{1s} \\ h_{2s} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1 - h_{1s} \\ h_2 - h_{2s} \end{bmatrix}, \quad u = F - F_s \]

• The element of the A (Jacobian) and B matrices

\[
A_{11} = \frac{\partial f_1}{\partial h_1} \bigg|_{h_s,F_s} = -\frac{R_1}{2A_1\sqrt{h_{1s} - h_{2s}}} \\
A_{12} = \frac{\partial f_1}{\partial h_2} \bigg|_{h_s,F_s} = \frac{R_1}{2A_1\sqrt{h_{1s} - h_{2s}}} \\
A_{21} = \frac{\partial f_2}{\partial h_1} \bigg|_{h_s,F_s} = \frac{R_1}{2A_2\sqrt{h_{1s} - h_{2s}}} \\
A_{22} = \frac{\partial f_2}{\partial h_2} \bigg|_{h_s,F_s} = -\frac{R_1}{2A_2\sqrt{h_{1s} - h_{2s}}} - \frac{R_2}{2A_2\sqrt{h_{2s}}} \\
B_{11} = \frac{\partial f_1}{\partial F} \bigg|_{h_s,F_s} = \frac{1}{A_1} \\
B_{21} = \frac{\partial f_2}{\partial F} \bigg|_{h_s,F_s} = 0
\]
• Only the height of the second tank is measured

\[ y = g(h_1, h_2, F) = h_2 - h_{2s} \]

\[ C_{11} = \left. \frac{\partial g}{\partial h_1} \right|_{h_s,F_s} = 0 \]

\[ C_{12} = \left. \frac{\partial g_2}{\partial h_2} \right|_{h_s,F_s} = 1 \]

• The state-space model is

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix} =
\begin{bmatrix}
- \frac{R_1}{2A_1 \sqrt{h_{1s} - h_{2s}}} & - \frac{R_1}{2A_1 \sqrt{h_{1s} - h_{2s}}} \\
\frac{R_1}{2A_2 \sqrt{h_{1s} - h_{2s}}} & - \frac{R_1}{2A_2 \sqrt{h_{1s} - h_{2s}}} - \frac{R_2}{2A_2 \sqrt{h_{2s}}}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
\frac{1}{A_1}
\end{bmatrix}[u]
\]

\[ y = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (y = x_2 = h_2 - h_{2s}) \]
Interpretation of Linearization

- Consider the single tank problem (assume $F$ is constant)

$$\frac{dh}{dt} = \frac{F}{A} - \frac{R}{A}\sqrt{h} = f(h, F) = 1 - \frac{1}{\sqrt{5}} \sqrt{h}$$

- Linearization $f(h, F) \approx 0 - \frac{1}{10}(h - h_s)$

The linear approximation works well between 3.5 to 7 feet.

The two functions are exactly equal at the steady-state value of 5 feet.
Exercise: interacting tanks

- Two interacting tank in series with outlet flowrate being function of the square root of tank height
  - Parameter values
    \[ R_1 = 2.5 \frac{\text{ft}^{2.5}}{\text{min}} \quad R_2 = \frac{5}{\sqrt{6}} \frac{\text{ft}^{2.5}}{\text{min}} \quad A_1 = 5 \text{ft}^2 \quad A_2 = 10 \text{ft}^2 \]
  - Input variable \( F = 5 \text{ ft}^3/\text{min} \)
  - Steady-state height values: \( h_{1s} = 10 \), \( h_{2s} = 6 \)

- Perform the following simulation using state-space model
  - What are the responses of tank height if the initial heights are \( h_1(0) = 12 \text{ ft} \) and \( h_2(0) = 7 \text{ ft} \) ?
  - Assume the system is at steady-state initially. What are the responses of tank height if
    - \( F \) changes from 5 to 7 \( \text{ ft}^3/\text{min} \) at \( t = 0 \)
    - \( F \) has periodic oscillation of \( F = 5 + \sin(0.2t) \)
    - \( F \) changes from 5 to 4 \( \text{ ft}^3/\text{min} \) at \( t = 20 \)
Stability of State-Space Models

• A state-space model is said to be stable if the response $x(t)$ is bounded for all $u(t)$ that is bounded.

• **Stability criterion for state-space model**
  - The state-space model will exhibit a bounded response $x(t)$ for all bounded $u(t)$, if and only if all of the eigenvalues of $A$ have **negative real parts**
    (the stability is independent matrices $B$ and $C$)

• Single variable equation $\dot{x} = ax$ has the solution
  $$x(t) = e^{at}x(0) \Rightarrow \text{stable if } a < 0$$

• The solution of $\dot{x} = Ax$ is $x(t) = e^{At}x(0)$
  - Stable if all of the eigenvalues of $A$ are less than zero
  - The response $x(t)$ is oscillatory if the eigenvalues are complex
Exercise

• Consider the following system equations

\[
\begin{align*}
\dot{x}_1 &= -0.5x_1 + x_2 \\
\dot{x}_2 &= 2x_2 \\
\end{align*}
\]

- Find the responses of \( x(t) \) for \( x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( x(0) = \begin{bmatrix} -0.5547 \\ 0.8321 \end{bmatrix} \)
  (slow subspace v.s. fast subspace)

• Consider the following system equations

\[
\begin{align*}
\dot{x}_1 &= 2x_1 + x_2 \\
\dot{x}_2 &= 2x_1 - x_2 \\
\end{align*}
\]

- Find the responses of \( x(t) \) for \( x(0) = \begin{bmatrix} 0.2703 \\ -0.9628 \end{bmatrix} \) and \( x(0) = \begin{bmatrix} 0.8719 \\ 0.4896 \end{bmatrix} \)
  (stable subspace v.s. unstable subspace)

Note: Find eigenvalue and eigenvector of \( A \)

\[
[V, D] = \text{eig}(A)
\]